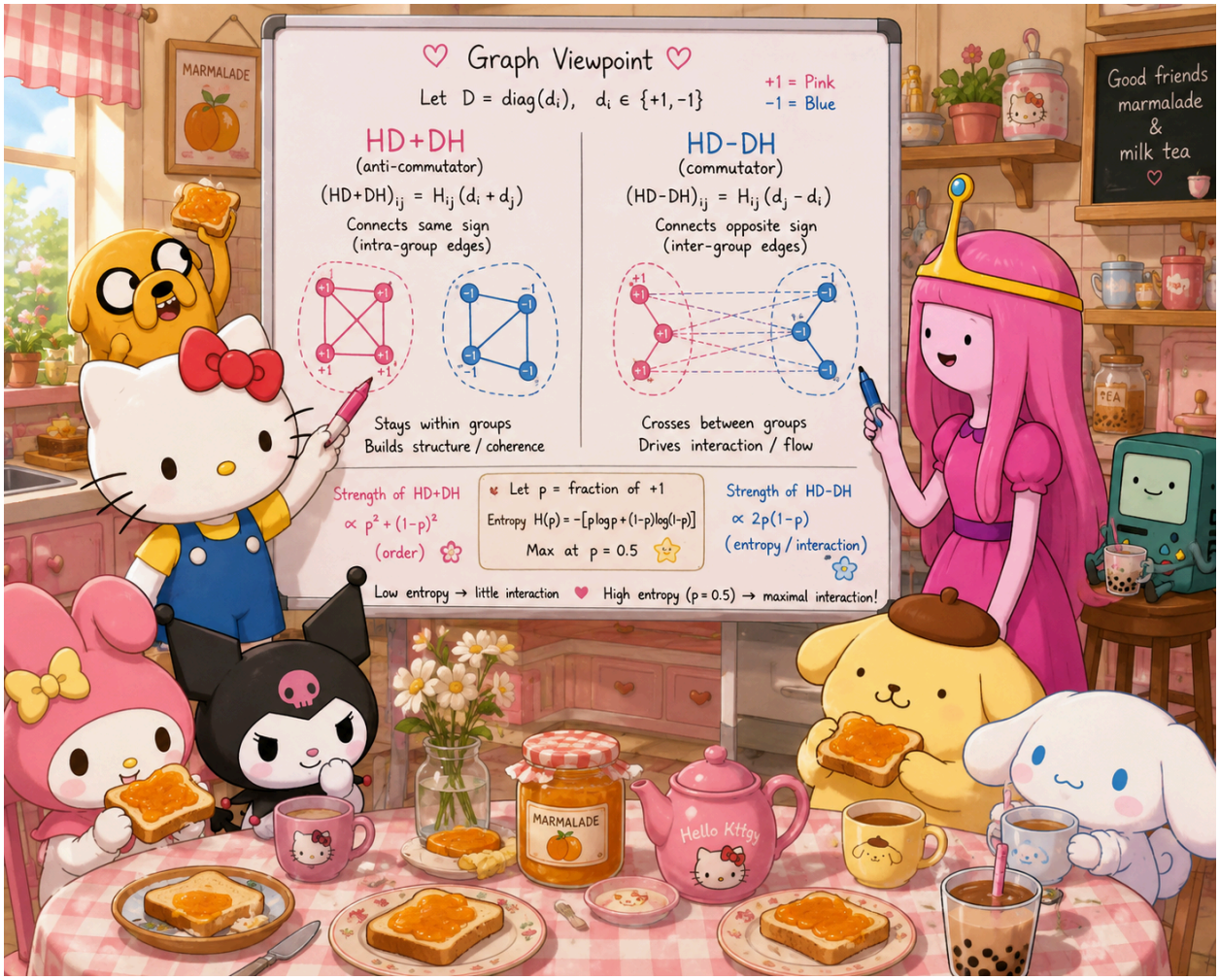

Core setup

Let $d_i \in \{+1, -1\}$ and define:

- X = HD+DH
- Y = HD−DH

Entries:

$X_{ij} = H_{ij}(d_i + d_j)$, $Y_{ij} = H_{ij}(d_j - d_i)$

So entries are always in {+2, 0, −2} (ignoring normalization).

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1) Extreme cases

D=I (all +1)

- X=2H → all entries ±2
- Y=0

👉 **Only agreement, no interaction**

D=−I (all −1)

- X=−2H → all entries ±2
- Y=0

👉 Same: **uniform system, no cross-effects**

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2) Mixed D (some +1, some −1)

For each pair (i,j):

If $d_i=d_j$ (agreement)

- $X_{ij} = \pm 2H_{ij}$
- $Y_{ij} = 0$

👉 **agreement → lives in X, killed in Y**

If $d_i \neq d_j$ (disagreement)

- $X_{ij}=0$
- $Y_{ij}=\pm 2H_{ij}$

👉 **disagreement → lives in Y, killed in X**

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3) Sign meaning in Y

When $d_i \neq d_j$:

- $Y_{ij}=+2H_{ij}$ if $d_j>d_i$
- $Y_{ij}=-2H_{ij}$ if $d_j<d_i$

👉 **direction encoded (who is + vs −)**

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4) Graph view

Think nodes colored by did_idi:

X=HD+DH

- edges only between **same-color nodes**
- weights = ±2

👉 **two disconnected clusters (within-group graph)**

Y=HD−DH

- edges only between **different-color nodes**
- weights = ±2 with direction

👉 **bipartite graph (between-group, directed)**

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5) One-line takeaway

- X: **agreement → clustering**
- Y: **disagreement → cross-group flow**